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A note on nonlinear acoustic resonances in rectangular cavities

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The problem of the resonant response of a gas contained in a two-dimensional rectangular cavity to periodic (sinusoidal) velocity excitations at the walls of the cavity is investigated. It is found that in some neighbourhood of each resonant frequency there are discontinuous pressure disturbances (shock waves). The present theory is an extension of Chester's theory on resonances in closed tubes.

1. Introduction

Following work by Chester (1964) and the present author (1975, 1976*a*, *b*, 1977), we investigate the resonant response of a gas contained in a two-dimensional rectangular cavity to vibrations of the walls. If the corners of the cavity are defined by

$$(x, y) = (0,0), (a,0), (0,b), (a,b)$$

we have the following acoustic eigenfrequencies:

$$\nu_{m,n} = \frac{1}{2\pi}\omega_{m,n} = \frac{c_0}{2}\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)^{\frac{1}{2}}. \tag{1}$$

For each of these eigenfrequencies there is a resonant field consisting of a superposition of the four waves

$$f_{\pm\pm}^{(m,n)} = f\left(t \pm \frac{x\cos\phi_{m,n} \pm y\sin\phi_{m,n}}{c_0}\right), \tag{2}$$

where

$$\phi_{m,n} = \arctan\left(\frac{a}{b}\,\frac{m}{n}\right).$$

For simplicity we assume that the vibration of the wall is sinusoidal in time. The problem is no more difficult for any other periodic time dependence of the wall oscillations. However, since the problem is nonlinear and therefore solutions cannot be directly superposed, a special assumption is inevitable. Without essential loss of generality we define the boundary conditions by

$$u(0,y,t) = 0, \quad v(x,0,t) = 0, \tag{3}$$
$$u(a,y,t) = u_W(y)\cos\omega t, \quad v(x,b,t) = v_W(x)\cos\omega t,$$

where u and v are the x and the y components of the particle velocity, respectively.

We use small amplitude expansions

$$\left.\begin{aligned} c_0+c &= c_0+c_1+c_2+\dots, \\ u &= \phantom{c_0+{}} u_1+u_2+\dots, \\ v &= \phantom{c_0+{}} v_1+v_2+\dots \end{aligned}\right\} \tag{4}$$

for the sound speed and the components of the particle velocity. In the interest of generality, we do not introduce powers of the Mach number M into these expansions, since the Mach number dependence of the different orders in (4) is not clear at this stage. This problem has been discussed by the author (1975, 1976*b*). Nevertheless the assumption of small amplitudes implies that $M \ll 1$. The first-order (acoustic) quantities can be expressed as follows:

$$\left.\begin{aligned} c_1 &= \tfrac{1}{8}(\gamma-1)\{f_{-+}+f_{++}+f_{--}+f_{+-}\}, \\ u_1 &= \tfrac{1}{4}\cos\phi\{f_{-+}-f_{++}+f_{--}-f_{+-}\}, \\ v_1 &= \tfrac{1}{4}\sin\phi\{f_{-+}-f_{++}-f_{--}+f_{+-}\}, \end{aligned}\right\} \tag{5}$$

where γ is the ratio of specific heats and, for convenience, the superscript (m,n) has been dropped. As the first-order velocity components do not satisfy the boundary conditions (3) at exact resonance, u and v have to be computed to second order.

2. Resonance equation and solutions

When the expressions (5) are inserted in the second-order terms of the Eulerian equations, the following equations can be derived (after considerable manipulation):

$$\Box\,(u_1+u_2) = \frac{\gamma+1}{32c_0}\cos\phi\,\frac{\partial^2}{\partial t^2}[f_{-+}^2+f_{--}^2-f_{++}^2-f_{+-}^2]+G(x,y,t), \tag{6}$$

$$\Box\,(v_1+v_2) = \frac{\gamma+1}{32c_0}\sin\phi\,\frac{\partial^2}{\partial t^2}[f_{-+}^2-f_{--}^2-f_{++}^2+f_{+-}^2]+D(x,y,t), \tag{7}$$

where
$$\Box \equiv \frac{\partial^2}{\partial t^2}-c_0^2\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\},$$

$$\begin{aligned} G(x,y,t) = {}& \frac{\cos\phi}{4c_0}\left[\frac{\gamma+1}{4}-\sin^2\phi\right][f_{-+}f_{--}-f_{++}f_{+-}]'' \\ &+\frac{3-\gamma}{16c_0}\cos\phi\{[f_{-+}+f_{--}][f_{++}+f_{+-}]'-[f_{++}+f_{+-}][f_{-+}+f_{--}]'\}' \\ &-\frac{\cos\phi\sin^2\phi}{4c_0}\{f_{-+}f_{+-}'-f_{+-}f_{-+}'-f_{++}f_{--}'+f_{--}f_{++}'\}', \end{aligned} \tag{8}$$

$$\begin{aligned} D(x,y,t) = {}& \frac{\sin\phi}{4c_0}\left[\frac{\gamma+1}{4}-\cos^2\phi\right][f_{-+}f_{+-}-f_{++}f_{--}]'' \\ &+\frac{3-\gamma}{16c_0}\sin\phi\{[f_{-+}+f_{+-}][f_{++}+f_{--}]'-[f_{++}+f_{--}][f_{-+}+f_{+-}]'\}' \\ &-\frac{\cos^2\phi\sin\phi}{4c_0}\{f_{-+}f_{--}'-f_{--}f_{-+}'-f_{++}f_{+-}'+f_{+-}f_{++}'\}' \end{aligned} \tag{9}$$

and the primes indicate derivatives with respect to time.

Integrating (6) and (7) leads to

$$u_2 = \frac{\gamma+1}{64c_0^2}\cos\phi\left\{[x\cos\phi+y\sin\phi]\frac{\partial}{\partial t}[f_{-+}^2+f_{++}^2] + [x\cos\phi-y\sin\phi]\frac{\partial}{\partial t}[f_{--}^2+f_{+-}^2]\right\}+A(x,y,t), \tag{10}$$

$$v_2 = \frac{\gamma+1}{64c_0^2}\sin\phi\left\{[x\cos\phi+y\sin\phi]\frac{\partial}{\partial t}[f_{-+}^2+f_{++}^2] - [x\cos\phi-y\sin\phi]\frac{\partial}{\partial t}[f_{--}^2+f_{+-}^2]\right\}+B(x,y,t). \tag{11}$$

A and B are lengthy expressions which can be obtained in a straightforward way from the relations

$$\Box f_{-+}f_{++} = 4f'_{-+}f'_{++}, \qquad \Box f_{-+}f_{--} = 4\sin^2\phi f'_{-+}f'_{--},$$
$$\Box f_{-+}f_{+-} = 4\cos^2\phi f'_{-+}f'_{+-}, \quad \Box f_{++}f_{--} = 4\cos^2\phi f'_{++}f'_{--},$$
$$\Box f_{++}f_{+-} = 4\sin^2\phi f'_{++}f'_{+-}, \quad \Box f_{--}f_{+-} = 4f'_{--}f'_{+-}.$$

It turns out that

$$A(0,y,t) = A(a,y,t) = 0, \quad B(x,0,t) = B(x,b,t) = 0 \tag{12}$$

when all the boundary conditions are satisfied. This is a remarkable result which has been pointed out before by Seymour & Mortell (1973) for a one-dimensional resonator. The functions A and B can be interpreted as terms which account for the interaction of the four waves. The physical meaning of (12) can then be explained as follows.

A wave path starting at a certain point on one of the walls is reflected once at each of the other walls before it comes back to the starting point. The effects of the wave field on the travelling time of a wavelet which travels along this path just cancel when the wavelet has gone through a full cycle.

The discussion of the solutions (10) and (11) is similar to that given by Chester (1964) and Keller (1976*b*). Here considerations are restricted to excitation functions (wall displacements) which are sinusoidal in time and exactly resonant. At the walls of the cavity the terms on the right-hand sides of (10) and (11), together with the first-order velocity components given by (5), have to be equated to the corresponding velocity components defined by the boundary conditions (3). Owing to (12), A and B do not contribute. It can easily be shown that this procedure leads to a resonance equation of the form

$$df^2(t)/dt = \epsilon^2\cos\omega t, \tag{13}$$

where ϵ is a constant which has to be determined later.

The velocity amplitudes $u_W(y)$ and $v_W(x)$ at the walls can be split into a pair of resonant excitation functions u_{WR} and v_{WR} and a pair of non-resonant (linear) excitation functions u_{WA} and v_{WA}:

$$u_W = u_{WR}+u_{WA}, \quad v_W = v_{WR}+v_{WA}. \tag{14}$$

Introducing (13) in (10) and (11) gives

$$u_2 = u_{WR} = \frac{\gamma+1}{16c_0^2}(-1)^n\epsilon^2 a\cos^2\phi\cos\left(m\pi\frac{y}{b}\right),$$
$$v_2 = v_{WR} = \frac{\gamma+1}{16c_0^2}(-1)^m\epsilon^2 b\sin^2\phi\cos\left(n\pi\frac{x}{a}\right). \tag{15}$$

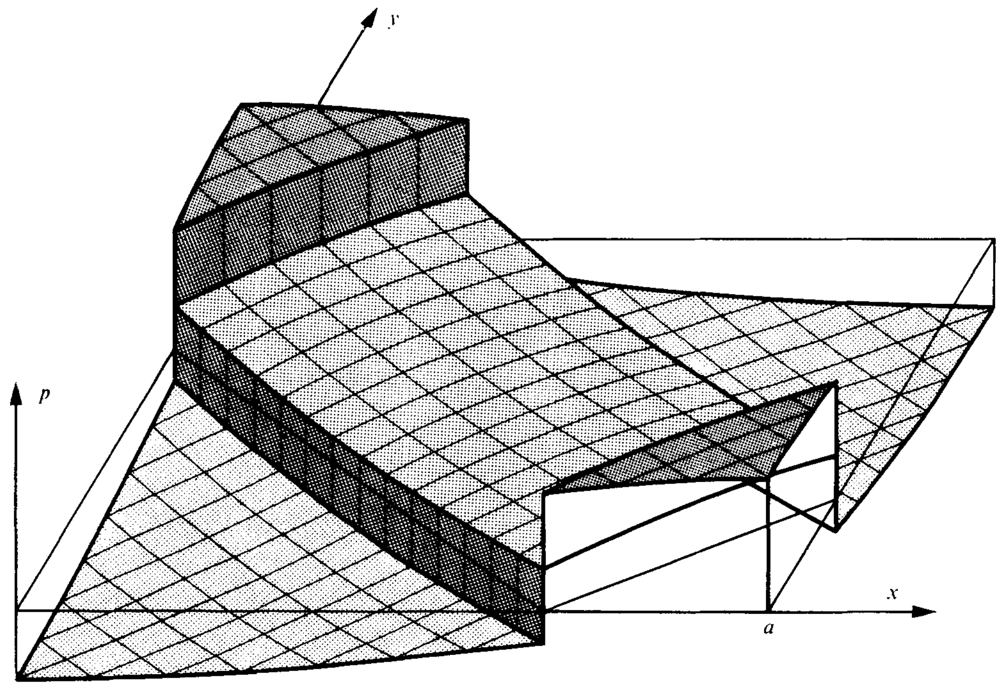

FIGURE 1. Typical solution for the resonant pressure disturbances. The pressure disturbance p is plotted *vs.* the spatial co-ordinates x and y for the mode (1, 1) at $\omega t = 0{\cdot}6\pi$.

Equation (13) can be recognized as the resonance equation which has been discussed extensively by Chester (1964). Having solved (13) we can easily deduce the first-order pressure disturbance

$$p_1(x,y,t) = \frac{\gamma}{4}\epsilon\left(\frac{2}{\omega}\right)^{\frac{1}{2}}\frac{p_0}{c_0}\left\{\operatorname{sc}\left(\frac{\omega t}{2}-\frac{\pi}{4}\left[2n\frac{x}{a}+2m\frac{y}{b}-1\right]\right)\right.$$
$$+\operatorname{sc}\left(\frac{\omega t}{2}+\frac{\pi}{4}\left[2n\frac{x}{a}+2m\frac{y}{b}+1\right]\right)$$
$$+\operatorname{sc}\left(\frac{\omega t}{2}-\frac{\pi}{4}\left[2n\frac{x}{a}-2m\frac{y}{b}-1\right]\right)$$
$$\left.+\operatorname{sc}\left(\frac{\omega t}{2}+\frac{\pi}{4}\left[2n\frac{x}{a}-2m\frac{y}{b}+1\right]\right)\right\}, \qquad (16)$$

where

$$\operatorname{sc}(\tau) \equiv \begin{Bmatrix} \cos\tau & \text{if} & \sin\tau > 0, \\ 0 & \text{if} & \sin\tau = 0, \\ -\cos\tau & \text{if} & \sin\tau < 0. \end{Bmatrix} \qquad (17)$$

A typical solution (for $m = n = 1$, $\omega t = 0{\cdot}6\pi$) is shown in figure 1.

With the help of (15) and (16) we can calculate the time-averaged power input $\langle W\rangle$:

$$\langle W\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\cos\omega t\left\{\int_0^a p_1(x,b,t)\,v_{WR}(x)\,dx+\int_0^b p_1(a,y,t)\,u_{WR}(y)\,dy\right\}dt$$
$$= \frac{\gamma(\gamma+1)}{12\pi c_0^2}ab\,\frac{\epsilon^3}{(2\omega)^{\frac{1}{2}}}\frac{p_0}{c_0}. \qquad (18)$$

In this equation v_{WR} and u_{WR} can be replaced by v_W and u_W [see remarks above (14)]. This leads to a new equation which can be compared with (19). Using the definitions (3), the comparison leads to the determination of ϵ:

$$\epsilon = \left\{\frac{32c_0^2}{(\gamma+1)\,ab}\left|(-1)^n\int_0^b u_W(y)\cos\left(m\pi\frac{y}{b}\right)dy+(-1)^m\int_0^a v_W(x)\cos\left(n\pi\frac{x}{a}\right)dx\right|\right\}^{\frac{1}{2}}. \quad (19)$$

From (14)–(19) it is easy to show that u_{WA} and v_{WA} do not contribute to $\langle W\rangle$ in the limit $M \rightarrow 0$ and the definition (3) is in no way restricted. However, if we require that M is small (but not infinitesimally small) it is clear that u_W and v_W should not be too 'strongly' different from u_{WR} and v_{WR} since otherwise the Mach-number expansion would become meaningless.

From the physical point of view we can argue as follows. If the time-averaged distribution of the power input along the walls of the cavity is considerably different from that produced by u_{WR} and v_{WR}, then the energy-redistribution mechanism dominates the problem. In this case we can think in terms of second-order waves running along the main wave fronts, thus distributing the energy (which is fed in at the walls) continuously over the whole wave field. This more difficult problem would essentially correspond to an extension to two-dimensional cavities of the theory on subharmonic resonances in closed tubes by the present author (1975). Note that in this case the Mach-number dependence of the orders in the expansions (4) would be different.

To include solutions for which the frequency is slightly different from a resonant frequency we could replace $f(t)$ in (10) and (11) by $f(t)+\alpha\Delta\omega$, where $\Delta\omega$ is the deviation from a certain resonant angular frequency and α is some suitable constant. As in case of a one-dimensional resonator there is a frequency range around every resonant frequency where the solutions are discontinuous if frictional damping is neglected. The width of these ranges where shock waves occur is $O(\Delta\omega/\omega) = O(M^{\frac{1}{2}})$. There the amplitudes of the disturbances are larger by a factor $O(M^{-\frac{1}{2}})$ than in the range where linear acoustic theory applies.

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